1a) Compute $G^{(2)}_{\varphi}(x_1, x_2)$ up to and including order g^2

$$\mathcal{L} = \mathcal{L}_0 + \mathcal{L}_I \qquad \text{with} \quad \mathcal{L}_I = g\,\overline{\psi}\psi\varphi$$

$$G^{(2)}_{\varphi}(x_1,x_2) = \frac{1}{Z_0}\int \mathcal{D}\varphi \int \mathcal{D}\overline{\psi}\mathcal{D}\psi \; e^{i\int d^4x \{\mathcal{L}_0 + \mathcal{L}_I\}} \varphi(x_1)\,\varphi(x_2)$$

with $Z_0 \equiv Z[J=0, \eta=\overline{\eta}=0, g=0]$

By expanding in powers of $\mathcal{L}_I$, equivalently, in powers of g, we obtain:

$$G^{(2)}_{\varphi}(x_1,x_2) = \frac{1}{Z_0}\int \mathcal{D}\varphi \int \mathcal{D}\overline{\psi}\mathcal{D}\psi \; e^{i\int d^4x\, \mathcal{L}_0} \times$$

$$\times \Big\{ \varphi(x_1)\varphi(x_2) + ig\int d^4w \; \overline{\psi}_\alpha(w)\psi_\alpha(w)\varphi(w)\,\varphi(x_1)\varphi(x_2)$$

$$+ \frac{1}{2!}(ig)^2 \int d^4w_1 \int d^4w_2 \; \overline{\psi}_\alpha(w_1)\psi_\alpha(w_1)\varphi(w_1)\,\overline{\psi}_\beta(w_2)\psi_\beta(w_2)\varphi(w_2)$$

$$\times\, \varphi(x_1)\varphi(x_2)$$

$$+ O(g^3)\Big\}$$

However, we note immediately that all contributions $O(g^{2n+1})$ (i.e., odd powers of g) vanish because they contain an odd number of φ fields. Hence:

$$G_\varphi^{(2)}(x_1, x_2) = \langle \varphi(x_1)\, \varphi(x_2) \rangle$$

$$-\frac{1}{2} g^2 \int d^4 w_1 \int d^4 w_2 \, \langle \bar{\psi}_\alpha(w_1)\, \psi_\alpha(w_1)\, \varphi(w_1)\, \bar{\psi}_\beta(w_2)\, \psi_\beta(w_2)\, \varphi(w_2)\, \varphi(x_1)\, \varphi(x_2) \rangle$$

$$+\, O(g^4)$$

$$= i D(x_1 - x_2) - \frac{1}{2} g^2 \int d^4 w_1 \int d^4 w_2 \, \Big\{$$

i) $$-2 \; iD(x_1 - w_1)\; iD(x_2 - w_2)\; iS_{\alpha\beta}(w_1 - w_2)\; iS_{\beta\alpha}(w_2 - w_1)$$

ii) $$+2 \; iD(x_1 - w_1)\; iD(x_2 - w_2)\; iS_{\alpha\alpha}(w_1 - w_1)\; iS_{\beta\beta}(w_2 - w_2)$$

iii) $$- \; iD(x_1 - x_2)\; iD(w_1 - w_2)\; iS_{\alpha\beta}(w_1 - w_2)\; iS_{\beta\alpha}(w_2 - w_1)$$

iv) $$+ \; iD(x_1 - x_2)\; iD(w_1 - w_2)\; iS_{\alpha\alpha}(w_1 - w_1)\; iS_{\beta\beta}(w_2 - w_2)$$

$$\Big\} + O(g^4)$$

where we employed:

$$\langle \varphi(x)\, \varphi(y) \rangle = \underbrace{\varphi(x)\, \varphi(y)} = iD(x - y)$$

$$\langle \psi_\alpha(x)\, \bar{\psi}_\beta(y) \rangle = \underbrace{\psi_\alpha(x)\, \bar{\psi}_\beta(y)} = iS_{\alpha\beta}(x - y)$$

with α, β spinor indices.
In the expression above we identify four contributions at order g^2, of which only term i) is connected, while ii), iii) and iv) are disconnected.

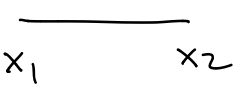
The factor 2 in i) and ii) is due to the exchange of w_1 and w_2 leading to equivalent Wick contractions.

The minus sign in i) and iii) is due to an odd number of permutations of the fermion fields ψ and $\bar{\psi}$.
Note that spinor indices α, β are summed over:

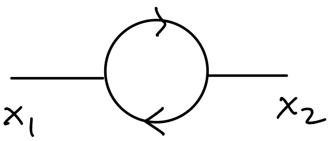
$$\begin{cases} i S_{\alpha\beta}(w_1 - w_2)\, i S_{\beta\alpha}(w_2 - w_1) = \mathrm{Tr}\left[i S(w_1 - w_2)\, i S(w_2 - w_1) \right] \\ i S_{\alpha\alpha}(w_1 - w_1) = \mathrm{Tr}\left[i S(w_1 - w_1) \right] \end{cases}$$

1b) The Feynman diagrams are as follows:

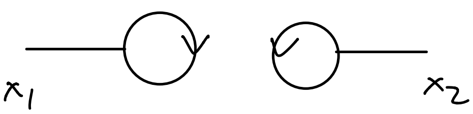
$O(g^0)$



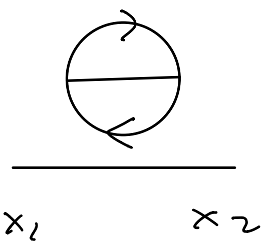
$O(g^2)$ i)



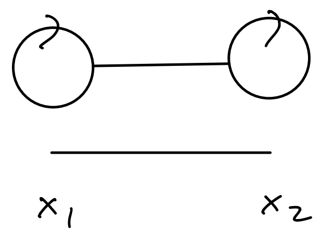
ii)



iii)

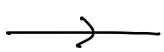


iv)



where

scalar



fermion

2a) The current $J_\mu = \bar{\psi} \gamma_\mu \psi$ is still conserved because $\mathcal{L}$ is still invariant under a $U(1)$ global transformation. Specifically, the new terms $\frac{1}{2} M^2 A_\mu A^\mu$ and $-\frac{1}{2\xi}(\partial_\mu A^\mu)^2$ are invariant because a $U(1)$ global transformation is:

$$\begin{cases} \psi'(x) = e^{i\alpha} \psi(x) \\ \bar{\psi}'(x) = e^{-i\alpha} \bar{\psi}(x) \\ A'_\mu(x) = A_\mu(x) \end{cases}$$

where α is not a function of spacetime and $A(x)$ does not change. *The fact that the fermion EoMs stay unchanged and $\partial_\mu J^\mu = 0$ stays true is also valid arg.

2b) EoM for A_μ, given the Lagrangian density:

$$\mathcal{L} = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} + \frac{1}{2} M^2 A_\mu A^\mu - \frac{1}{2\xi}\left(\partial_\mu A^\mu\right)^2 + \bar{\psi}(i\partial\!\!\!/ + eA\!\!\!/ - m)\psi$$

$$\frac{\delta \mathcal{L}}{\delta \partial_\mu A_\nu} = -\frac{1}{4} F^{\mu\nu} \cdot 4 - \frac{1}{2\xi} 2 g^{\mu\nu}\left(\partial_\lambda A^\lambda\right)$$

$$= - F^{\mu\nu} - \frac{1}{\xi} g^{\mu\nu}\left(\partial_\lambda A^\lambda\right)$$

$$\partial_\mu \frac{\delta \mathcal{L}}{\delta \partial_\mu A_\nu} = - \partial_\mu F^{\mu\nu} - \frac{1}{\xi} \partial^\nu \left(\partial_\lambda A^\lambda\right)$$

$$\frac{\delta \mathcal{L}}{\delta A_\nu} = \frac{1}{2} M^2 \cdot 2 A^\nu + e \bar{\psi} \gamma^\nu \psi$$

$$= M^2 A^\nu + e J^\nu$$

Thus the EoM reads:

$$- \partial_\mu F^{\mu\nu} - \frac{1}{\xi} \partial^\nu \left(\partial_\lambda A^\lambda\right) = M^2 A^\nu + e J^\nu$$

Equivalently:

$$\partial_\mu F^{\mu\nu} + \frac{1}{\xi} \partial^\nu (\partial \cdot A) + M^2 A^\nu = -e J^\nu$$

2c) Taking the divergence we obtain:

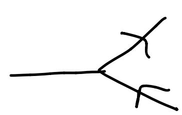
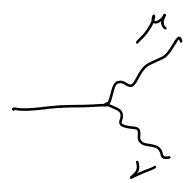
$$\partial_\nu \partial_\mu F^{\mu\nu} + \frac{1}{\xi} \partial_\nu \partial^\nu (\partial \cdot A) + M^2 \partial_\nu A^\nu = -e \partial_\nu J^\nu$$

The antisymmetry of $F^{\mu\nu}$ implies that the first term in the lhs vanishes:

$$\partial_\nu \partial_\mu F^{\mu\nu} = \frac{1}{2} \{ \partial_\mu, \partial_\nu \} \frac{1}{2} \left(F^{\mu\nu} - F^{\nu\mu} \right) = 0$$

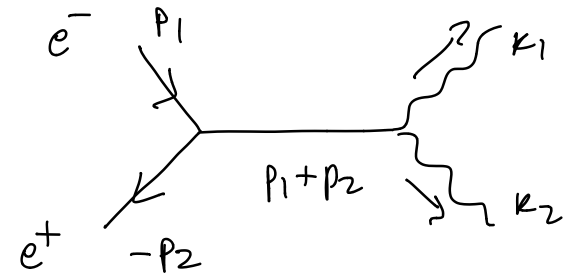
and employing $\partial_\nu J^\nu = 0$ we obtain:

$$\left(\frac{1}{\xi} \Box + M^2 \right) \partial \cdot A = 0$$

3a)  $i\lambda$ 

$2ig\, g^{\mu\nu}$

$e^+ e^- \to \gamma\gamma$ at tree level is given by the following Feynman diagram:



$$s = (p_1+p_2)^2$$

$$A = \frac{i}{(p_1+p_2)^2 - M^2} (if)(2ig)\, g_{\mu\nu}\, \varepsilon^{\mu}_{r'}(k_1)\, \varepsilon^{\nu}_{s'}(k_2) \times \left(\bar{v}_r(p_2)\, u_s(p_1)\right)$$

$$A^{+} = \frac{-i}{(p_1+p_2)^2 - M^2} (-if)(-2ig)\, g_{\mu\nu}\, \varepsilon^{\mu}_{r'}(k_1)\, \varepsilon^{\nu}_{s'}(k_2) \times \left(\bar{u}_s(p_1)\, v_r(p_2)\right)$$

Thus the unpolarized squared amplitude is:

$$X = \frac{1}{4} \sum_{r,s,r',s'} A A^{+}$$

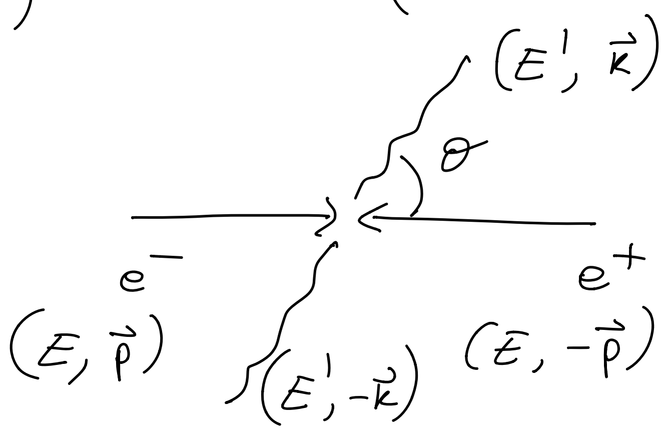
$$= \frac{1}{4} \frac{4 f^2 g^2}{(s - M^2)^2} \sum_{r,s=1}^{2} \left(\bar{v}_r(p_2)\, u_s(p_1)\right)\left(\bar{u}_s(p_1)\, v_r(p_2)\right) \times \sum_{r',s'=1}^{3} \varepsilon^{\mu}_{r'}(k_1)\, \varepsilon_{\mu s'}(k_2)\, \varepsilon^{\nu}_{r'}(k_1)\, \varepsilon_{\nu s'}(k_2)$$

$$X = \frac{f^2 g^2}{(s - M^2)^2} \quad \mathrm{Tr}\left[(\not{p}_1 + m)(\not{p}_2 - m)\right] \times$$

$$\times \left(-g_{\mu\nu} + \frac{k_{1\mu} k_{1\nu}}{M_A^2}\right)\left(-g^{\mu\nu} + \frac{k_2^\mu k_2^\nu}{M_A^2}\right)$$

$$= \frac{f^2 g^2}{(s - M^2)^2} \left(4 p_1 \cdot p_2 - 4m^2\right)\left(4 - \frac{k_1^2}{M_A^2} - \frac{k_2^2}{M_A^2} + \frac{(k_1 \cdot k_2)^2}{M_A^4}\right)$$

3b)



CM

$E = E'$

$$p_1 + p_2 = k_1 + k_2 \qquad p_1^2 = p_2^2 = m^2$$

$$k_1^2 = k_2^2 = M_A^2$$

$$s = (p_1 + p_2)^2 = 2m^2 + 2 p_1 \cdot p_2$$

$$s = (k_1 + k_2)^2 = 2M_A^2 + 2 k_1 \cdot k_2$$

Hence :

$$p_1 \cdot p_2 = \frac{s}{2} - m^2$$

$$k_1 \cdot k_2 = \frac{s}{2} - M_A^2$$

$$X = \frac{\ell^2 g^2}{(s-M^2)^2}\, 4 \left(\frac{s}{2} - 2m^2\right)\left(2 + \left(\frac{s}{2} - M_A^2\right)^2 \frac{1}{M_A^4}\right)$$

$$= \frac{2\ell^2 g^2}{(s-M^2)^2} \left(s - 4m^2\right) \left(2 + \frac{1}{4}\frac{\left(s-2M_A^2\right)^2}{M_A^4}\right)$$

$$= \frac{4\ell^2 g^2}{(s-M^2)^2} \left(s-4m^2\right) \left(1 + \frac{1}{2}\left(1 - \frac{s}{2M_A^2}\right)^2\right)$$

which is indeed written in terms of s, m, M and M_A.